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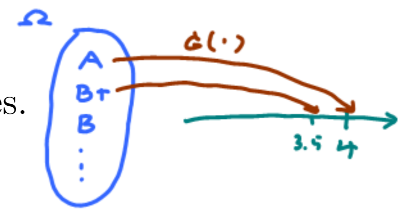
ECS315 2018/1 Part III.1 Dr.Prapun

deviates

7 Random variables

In performing a chance experiment, one is often not interested in the particular outcome that occurs but in a specific numerical value associated with that outcome. In fact, for most applications, measurements and observations are expressed as numerical quantities.

Example 7.1. Take this course and observe your grades.

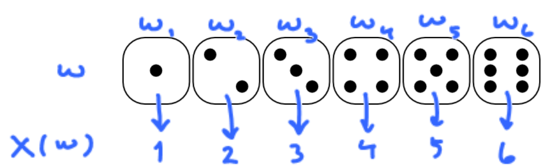


$$\Omega = \{\overset{\omega_1}{A}, \overset{\omega_2}{B+}, \overset{\omega_3}{B}, \overset{\omega_4}{C+}, \overset{\omega_5}{C}, D+, D, F\}$$

Define a function $G(\cdot)$ that maps the letter grades to numerical values:

$$G(A) = 4, G(B+) = 3.5, G(B) = 3, G(C+) = 2.5, G(C) = 2,$$
$$G(D+) = 1.5, G(D) = 1, G(F) = 0.$$

Example 7.2. Roll a dice. Let X be the result.



7.3. The advantage of working with numerical quantities is that we can perform mathematical operations on them.

↳ e.g. add, subtract, multiply, division, square, exp., power, log, average, average deviation from the mean, max, min

In the mathematics of probability, averages are called expectations or expected values.

7.4. *Intuitively*, a random variable is a "variable" that "takes on its values by chance".

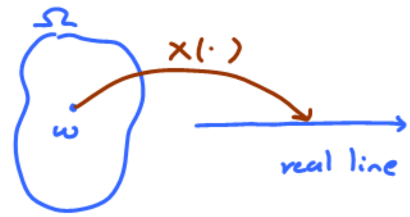
7.5. The convention is to use capital letters such as X, Y, Z to denote random variables.

Definition 7.6. A real-valued function $X(\omega)$ defined for all points ω in a sample space Ω is called a ***random variable*** (RV) [29].

- A random variable is a rule that assigns a numerical value to each possible outcome of a chance experiment.

$f : D \to R$ — f: a function; D: domain; R: range, codomain

$X : \Omega \to \mathbb{R}$ — Ω: sample space; $\mathbb{R}$: set of all real numbers



Example 7.7. Roll a fair dice:

$$\Omega = \{1, 2, 3, 4, 5, 6\} = \{\omega_i : \quad \omega_i = i, \quad i = 1, 2, \ldots, 6\}.$$

High school function: $y = f(x)$

Random variable: $x = X(\omega)$ — x: lower-case; X: upper-case

$X(\omega) = \omega$

$Y(\omega) = (\omega - 3)^2$

$Z(\omega) = \sqrt{\omega} = \sqrt{X(\omega)}$

$U(\omega) = \begin{cases} 1, & \omega > 3, \\ 0, & \omega \le 3 \end{cases}$

Observation:

(a) More than one random variables can be defined on one sample space.

(b) Although the function X, Y, Z, and U are deterministically defined, their values depend on the value of the outcome from the experiment, which is random.

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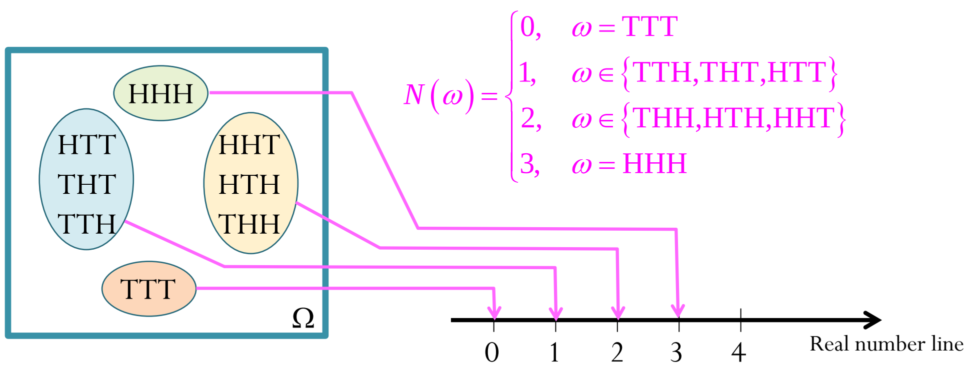
[29] The term "random variable" is a misnomer. Technically, if you look at the definition carefully, a random variable is a deterministic function; that is, it is not random and it is not a variable. [Toby Berger][25, p 254]

- As a function, it is simply a rule that maps points/outcomes ω in Ω to real numbers.
- It is also a deterministic function; nothing is random about the mapping/assignment. The randomness in the observed values is due to the underlying randomness of the argument of the function X, namely the experiment outcomes ω.
- In other words, the randomness in the observed value of X is induced by the underlying random experiment, and hence we should be able to compute the probabilities of the observed values in terms of the probabilities of the underlying outcomes.

Example 7.8 (Three Coin Tosses)**.** Counting the number of heads in a sequence of three coin tosses.

$$\Omega = \{\text{TTT},\text{TTH},\text{THT},\text{THH},\text{HTT},\text{HTH},\text{HHT},\text{HHH}\}$$



Example 7.9 (Sum of Two Dice)**.** If S is the sum of the dots when rolling one fair dice twice, the random variable S assigns the numerical value $i+j$ to the outcome (i,j) of the chance experiment.

Example 7.10. Continue from Example 7.7, Roll a fair dice

Let $X(\omega) = \omega$

$Y(\omega) = (\omega - 3)^2$

$\Omega = \{1, 2, \dots, 6\}$

$P(\{\omega\}) = \frac{1}{6} \quad \omega = 1, 2, \dots, 6$

(a) What is the probability that $X = 4$?

This actually means "$X(\omega) = 4$"

$X(\omega) = 4$ occurs when $\omega = 4$

Therefore, $P[X=4] = P(\{\underset{\omega}{4}\}) = \frac{1}{6}$

(b) What is the probability that $Y = 4$?

$Y(\omega) = 4$ occurs when $(\omega - 3)^2 = 4$

$\omega - 3 = \pm 2$

$\omega = 3 \pm 2 = 1$ or 5

Therefore, $P[Y=4] = P(\{1\}) + P(\{5\}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

Problem 11. (9 pt) Consider a sample space $\Omega = \{1, 3, 4\}$. Suppose, for $\omega = 1, 3, 4$, we have

$$P(\{\omega\}) = c\omega$$

for some constant c.

ω	$P(\{\omega\})$		
1	$c \times 1$	$= 1/8$	
3	$c \times 3$	$= 3/8$	
4	$c \times 4$	$= 4/8$	$= 1/2$

(a) (2 pt) Check that $c = 1/8$.

$P(\Omega) = 1$ $\quad c + 3c + 4c = 1$

$c = 1/8$

(b) (4 pt) Define $A = \{1, 3\}$ and $B = \{1, 4\}$.

(i) Find $P(A)$.

(ii) Find $P(A \cap B)$.

(c) (3 pt) Define a random variable X by $X(\omega) = \frac{12}{\omega}$.

ω	$X(\omega)$
1	$12/1 = 12$
3	$12/3 = 4$
4	$12/4 = 3$

(i) (1 pt) What are the possible values of X?

3, 4, 12

(ii) (1 pt) Find $P[X = 3]$.

$X(\omega) = 3$ when $\omega = 4$

$\Rightarrow P[X = 3] = P(\{4\}) = \frac{1}{2}$.

(iii) (1 pt) Find $P[X > 3]$.

$X(\omega) > 3$ when $\omega = 1, 3$

$\Rightarrow P[X > 3] = P(\{1\}) + P(\{3\}) = \frac{1}{8} + \frac{3}{8} = \frac{4}{8} = \frac{1}{2}$

Definition 7.11. Events involving random variables: When we write

or statement(s) about X

$$[\text{some condition(s) on } X],$$

event

we mean "the set of outcomes in Ω such that $X(\omega)$ satisfies the condition(s) specified."

- $[X = x] = \{\omega \in \Omega : X(\omega) = x\}$
  - We usually use the corresponding lowercase letter[30] to denote
    (a) a possible value (realization) of the random variable
    (b) the value that the random variable takes on
    (c) the running values for the random variable
- $[X \in B] = \{\omega \in \Omega : X(\omega) \in B\}$ Turn out this one is the most important one.
- $[a \le X < b] = \{\omega \in \Omega : a \le X(\omega) < b\}$
- $[X > a] = \{\omega \in \Omega : X(\omega) > a\}$

All of the above items are sets of outcomes. They are all events!

Example 7.12. Continue from Examples 7.7 and 7.10,

(a) $[X = 4] = \{\omega \in \Omega : X(\omega) = 4\}$

(b) $[Y = 4] = \{\omega \in \Omega : Y(\omega) = 4\} = \left\{\omega : (\omega - 3)^2 = 4\right\}$

7.13. Event of the form "[some condition(s) on X]" or "[some statement(s) about X]" can be written in the from $[X \in B]$ for some appropriate B.

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[30]This is the same as writing $[X = c]$ where c is a constant. Basically, it is a generic notation for $[X = 5]$, $[X = 1.6]$, $[X = \pi]$, etc. We use this when

(a) we don't want to specify the constant in the expression yet or

(b) we want to say that the statement/equation/property containing it is valid for any value of c.

It turns out that, later on, we will have to deal with many random variables and hence it is convenient to have the name of the constant c match the name of the corresponding random variable. So, we talk about the events $[X = x]$, $[Y = y]$, and $[Z = z]$ instead of having to find new name for the constant corresponding to each one of them, say, $[X = c]$, $[Y = d]$, and $[Z = h]$.

You may think we can use constants $c_1, c_2, \ldots$. However, we also will have to deal with ransom variables $X_1, X_2, \ldots, Y_1, Y_2, \ldots, Z_1, Z_2, \ldots$. So, again, will have to come up with new names for a lot of constants.

Example 7.14. Express each event below in the form $[X \in B]$.

(a) $[5 \le X < 8]$ $= [X \in [5,8)]$ $B = [5,8)$

(b) $[|X| < 3]$ $= [X \in (-3,3)]$ $B = (-3,3)$

(c) $[X > 2]$ $= [X \in (2,\infty)]$ $B = (2,\infty)$

(d) $[X = 1]$ $= [X \in \{1\}]$ $B = \{1\}$

Definition 7.15. To avoid double use of brackets (round brackets over square brackets), we write $P[X \in B]$ when we means $P([X \in B])$. Hence,

$$P[X \in B] \equiv P([X \in B]) = P(\{\omega \in \Omega : X(\omega) \in B\}).$$

Similarly,

$$P[X < x] = P([X < x]) = P(\{\omega \in \Omega : X(\omega) < x\}).$$

Definition 7.16. We also have another notation for $P[X \in B]$:

$$P^X(B) \equiv P[X \in B].$$

Observe that this function P^X is a set function. It maps subsets of real numbers into their probability values. Technically, we call this function the **law** or **distribution** of the random variable X. However, later on, we shall see that there are many functions that are also referred to as the "distribution" of X as well. They are all equivalent in the sense that they (almost surely) give the same information about probability concerning X.

Example 7.17. In Example 7.8 (Three Coin Tosses), if the coin is fair, then

$P[N < 2] \equiv P([N<2]) = P(\{TTT, HTT, THT, TTH\})$

$= P(\{TTT\}) + P(\{HTT\}) + P(\{THT\}) + P(\{TTH\}) = \frac{4}{8} = \frac{1}{2}$

#H

$[N<2] = \{\omega : N(\omega) < 2\} = \{TTT, HTT, THT, TTH\}$

7.18. Summary: In Chapter 5, we studied how to find the probability of any event A by adding the probabilities of the individual outcomes inside A. For example,

$$P(\{a, b, c\}) = P(\{a\}) + P(\{b\}) + P(\{c\}).$$

In this chapter, we now have steps to find any probability involving a random variable when the random variable is explicitly defined as a function of outcomes.

Step 1: Identify the sample space Ω and the probability for each outcome.

Step 2: Find the value of ω that makes $X(\omega)$ satisfy the given condition.

- For example, if we want to find the probability that $X = 3$, we need to find all $\omega \in \Omega$ that make $X(\omega) = 3$.
- The collection of such ω is the event that you are interested in.
- This basically turns the calculation into the one we know how to solve from Chapter 5.

Step 3: Find the probability of the event above.

- For example,

$$P(\{\omega_1, \omega_2, \omega_3\}) = P(\{\omega_1\}) + P(\{\omega_2\}) + P(\{\omega_3\}).$$

7.19. At a certain point in most probability courses, the sample space is rarely mentioned anymore and we work directly with random variables. The sample space often "disappears" along with the "(ω)" of $X(\omega)$ but they are really there in the background.

Definition 7.20. A set S is called a ***support*** of a random variable X if $P[X \in S] = 1$.

- To emphasize that S is a support of a particular variable X, we denote a support of X by S_X.
- Practically, we usually define a support of a random variable X to be the set of all the "possible" values of X.[31]
- For any random variable, the set $\mathbb{R}$ of all real numbers is always its support; however, it is not quite useful because it does not further limit the possible values of the random variable.
- Recall that a support of a probability measure P is any set $A \subset \Omega$ such that $P(A) = 1$.

Definition 7.21. The ***probability distribution*** is a description of the probabilities associated with the random variable.

7.22. There are three types of of random variables. The first type, which will be discussed in Section 8, is called ***discrete random variable***. In practice, to tell whether a random variable is discrete, one simple way is to consider the "possible" values of the random variable. If it is limited to only a finite or countably infinite number of possibilities, then it is discrete. We will later discuss ***continuous random variables*** whose possible values can be anywhere in some intervals of real numbers.

[31] Later on, you will see that 1) a default support of a discrete random variable is the set of values where the pmf is strictly positive and 2) a default support of a continuous random variable is the set of values where the pdf is strictly positive.

Ex. Back to Example 7.7 (fair dice)

$\Omega = \{1,2,3,4,5,6\}$

$Y(\omega) = (\omega-3)^2$

ω	$Y(\omega)$
1	4
2	1
3	0
4	1
5	4
6	9

(i) Consider the set $B_1 = \{0,1,4\}$.

Is B_1 a support of Y?

First find $P[Y \in B_1] = P(\{1,2,3,4,5\}) = \frac{5}{6} \neq 1$

So, set B_1 is <u>not</u> a support of Y.

(ii) Consider the set $B_2 = \{0,1,4,9\}$.

Is B_2 a support of Y?

First find $P[Y \in B_2] = P(\{1,2,3,4,5,6\}) = 1$

So, set B_2 is a support of Y.

(iii) Consider the set $B_3 = \{0,1,2,3,4,9\}$.

Is B_3 a support of Y?

First find $P[Y \in B_3] = P(\{1,2,3,\dots,6\}) = 1$

So, set B_3 is a support of Y.

Observation: (a) A RV can have multiple supports.

(b) The set $\mathbb{R}$ is always a support of any RV.

(c) Some supports contain "useless" members.

For example, in (iii) above,

we don't need "2" and "3" in B_3

to a support of Y.

Usually, we want to get the "minimal" support.

There are three kinds of RVs
- discrete RV (countable) (digital)
- continuous RV
- hybrid (mixed)